# Structural Iterative Rounding for Generalized $k$-Median Problems

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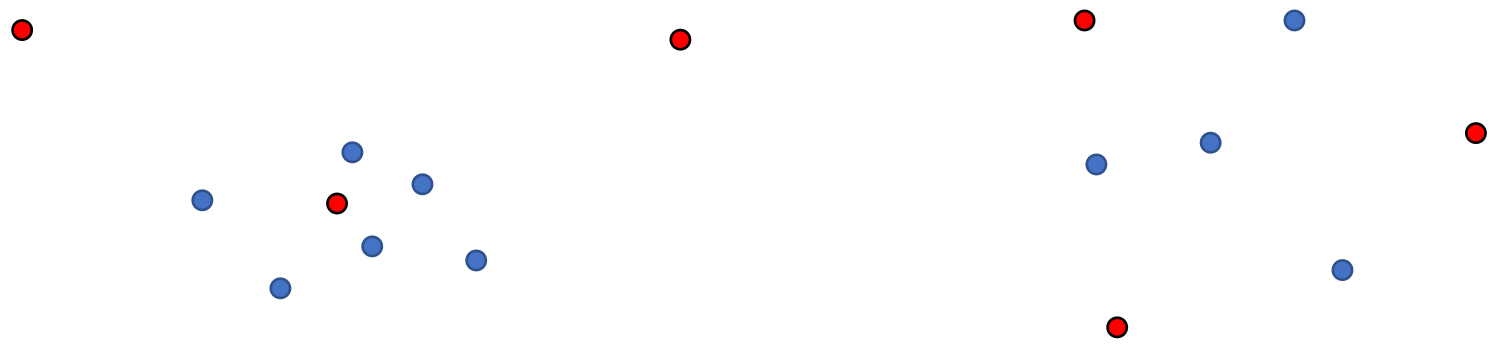

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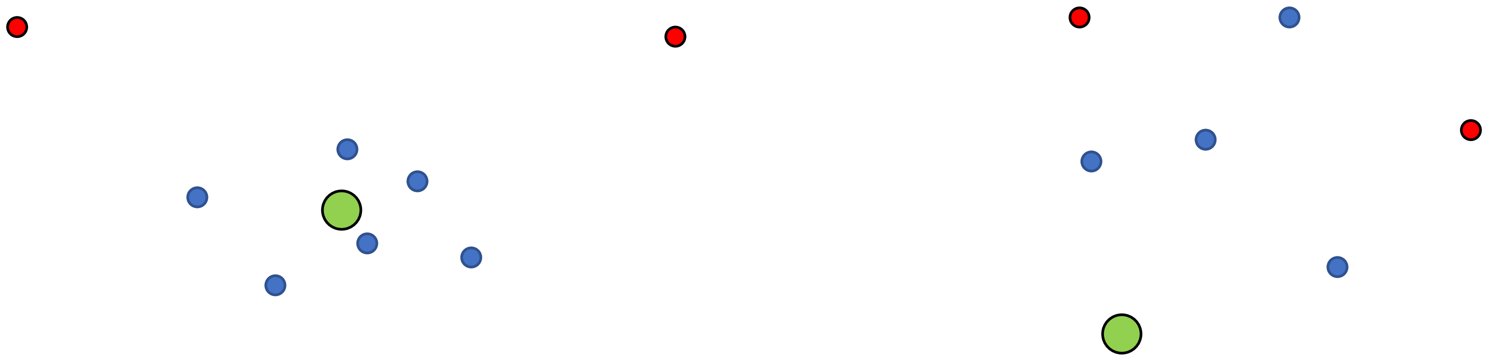
  - $F$ = set of Facilities
  - $k$ = # of Facilities to open



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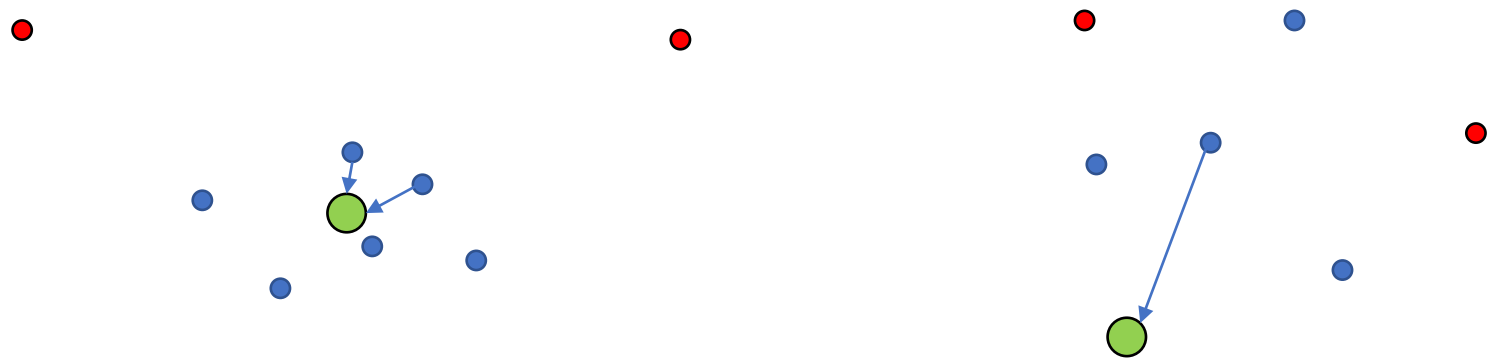
**Objective:**

minimize: $\sum_{j \in C} dist(j, nearest\ open\ facil.)$



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**$k$-Median with Outliers:** serve all clients ⇒ choose $m$ clients to serve

Best approx. for both is $\sim\mathbf{7}$ via Iterative Rounding (KLS, 2018)

*Ravishankar Krishnaswamy, Shi Li, Sai Sandeep:*
*Constant approximation for k-median and k-means with outliers via iterative rounding. STOC 2018: 646-659*

# Improving on KLS: Our Results

- **Knapsack Median:** $\sim 6.3$
- $\boldsymbol{k}$**-Median with Outliers:** $\sim 6.9$
- $\boldsymbol{k}$ $-$**Median with** $\boldsymbol{O}(\mathbf{1})$ **Side Constraints:** $\sim$6.3 (pseudo-approx.)

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# Basic LP: $k$-Median with Outliers

$$\min \sum_{i \in F, j \in C} d(i,j)\, x_{ij}$$

$x_{ij} \leq y_i \qquad \forall i \in F, j \in C$

$\sum_{i \in F} x_{ij} \leq 1 \qquad \forall j \in C$

$\sum_{i \in F, j \in C} x_{ij} \geq m$

$\sum_{i \in F} y_i \leq k$

$0 \leq x, y \leq 1$

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$x_{ij}$ ~ connect client $j$ to facil. $i$

$y_i$ ~ open facil. $i$

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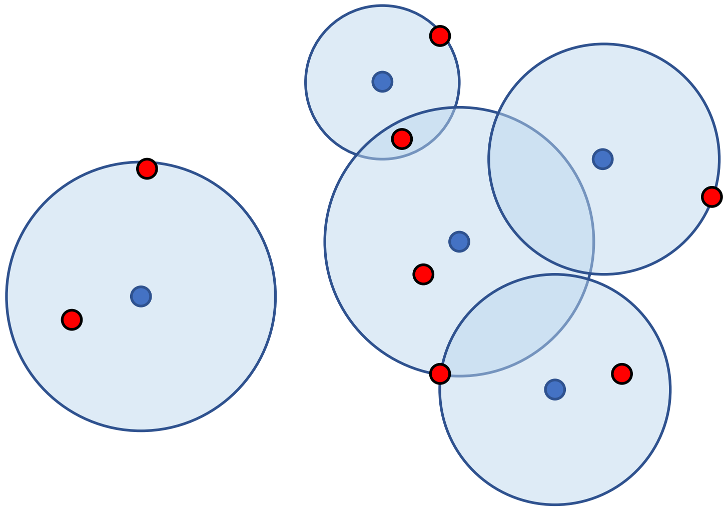
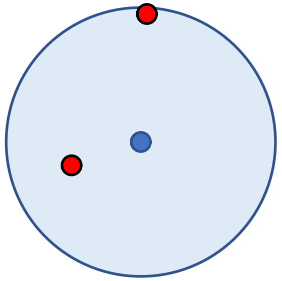


# ‘Set Cover’ LP: $k$-Median with Outliers



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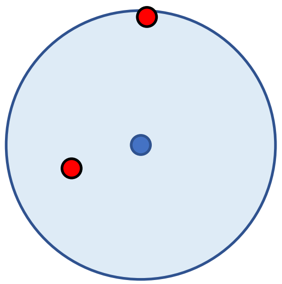

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$Ball_j = Client\ j's\ allowed\ connections$

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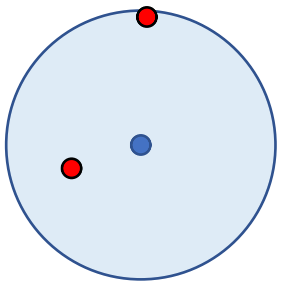

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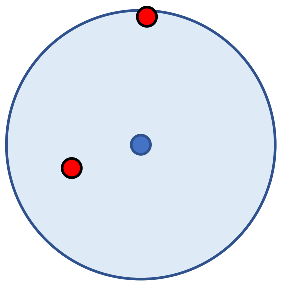

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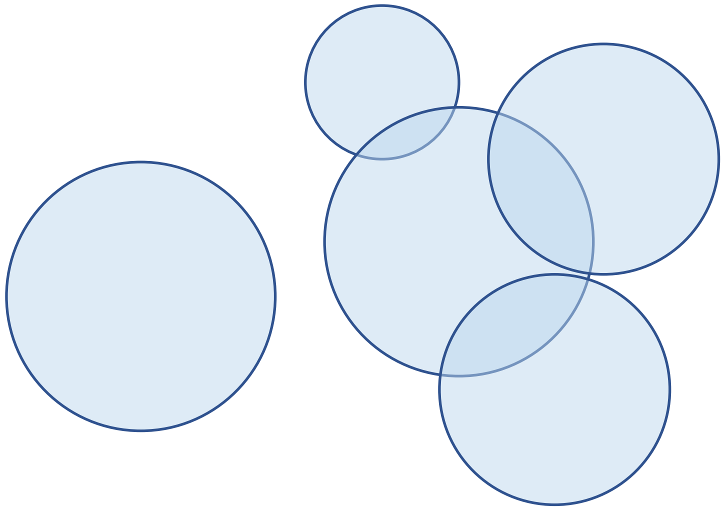
Cover $m$ Balls with $k$ open facils.



# Iterative Rounding Framework

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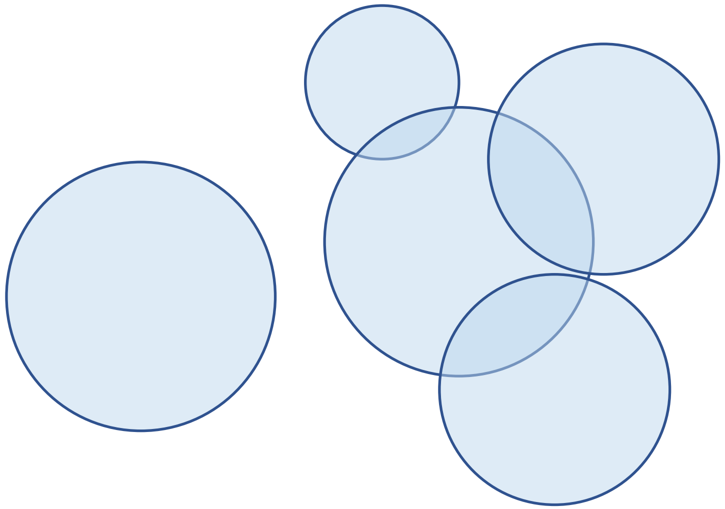

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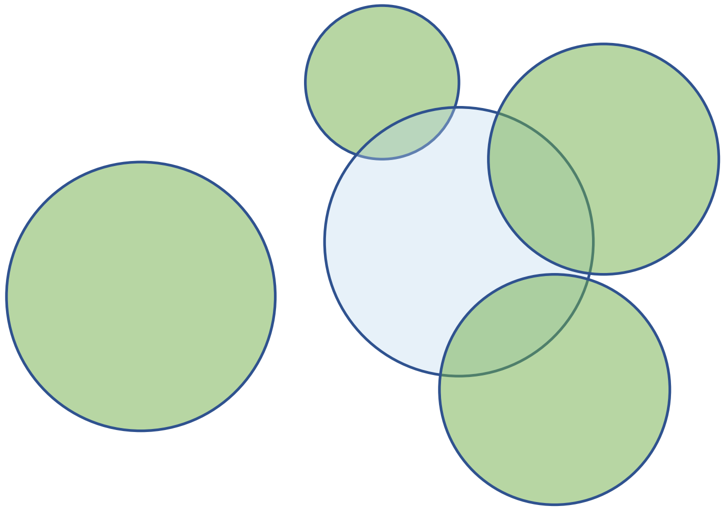

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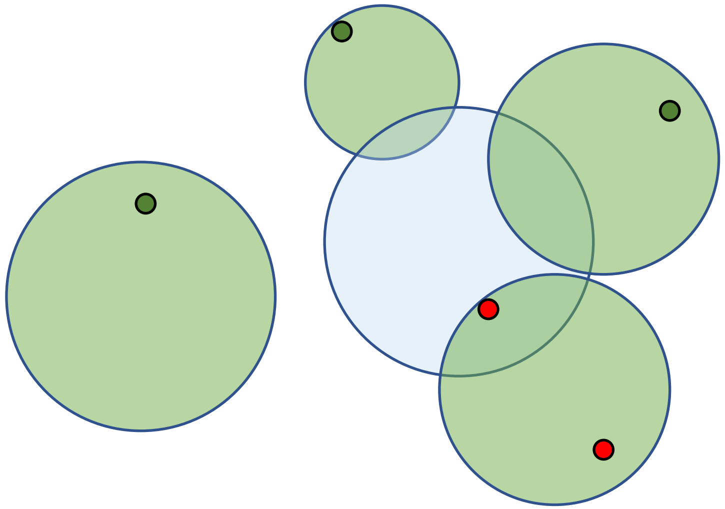

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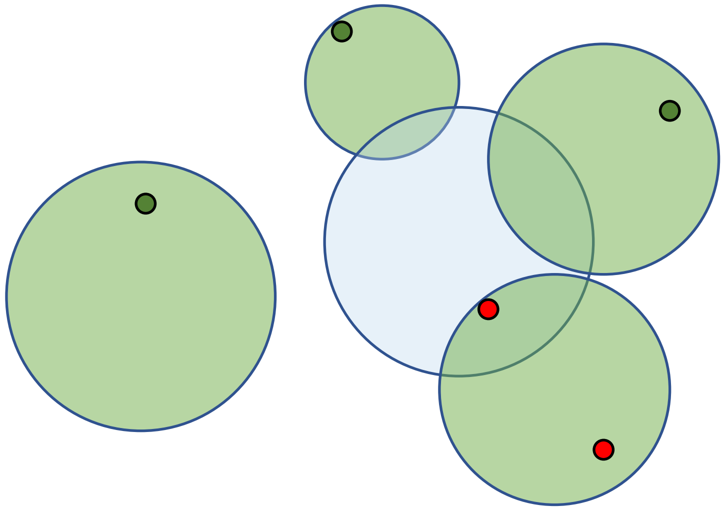
**Main Idea:** Control tight balls



# Improving KLS: Our Techniques

- KLS:
  - Desired Structure = disjoint balls
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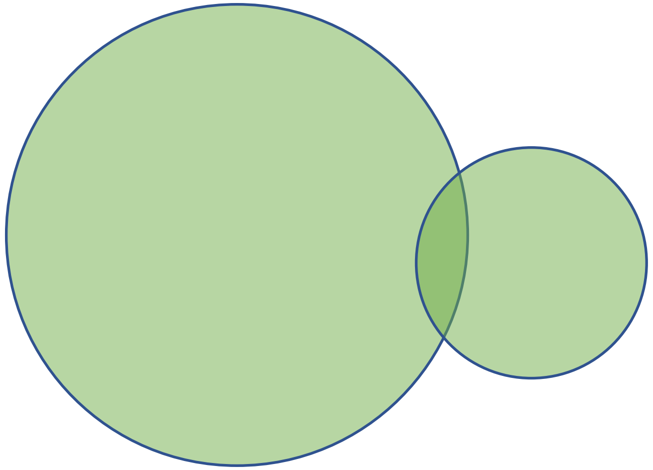

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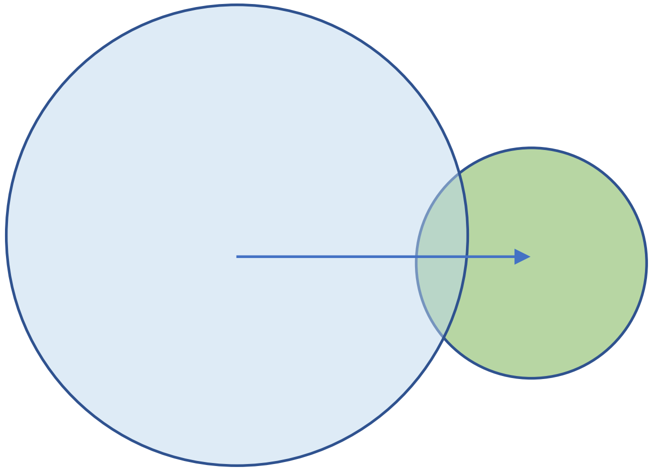

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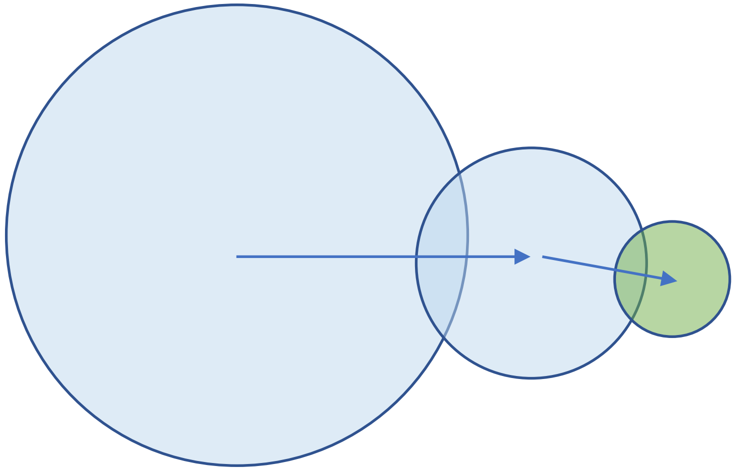

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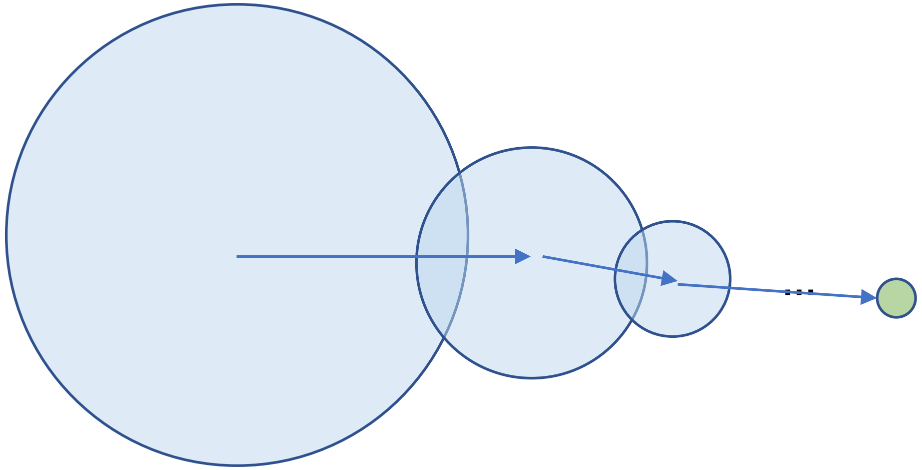


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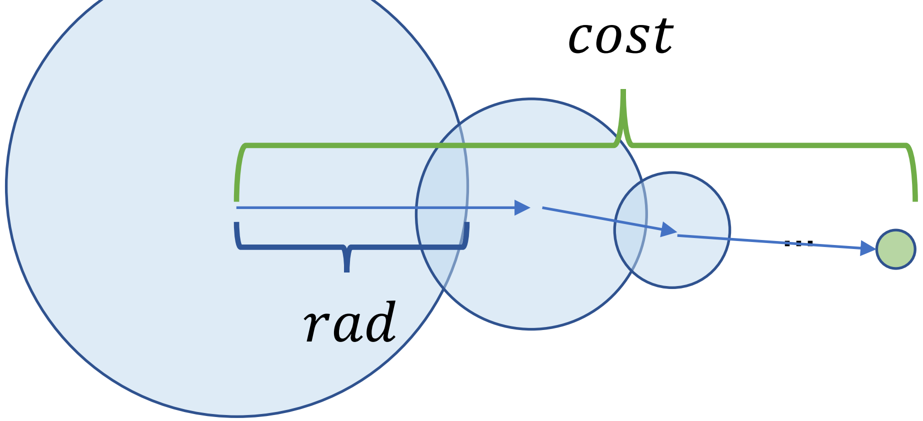

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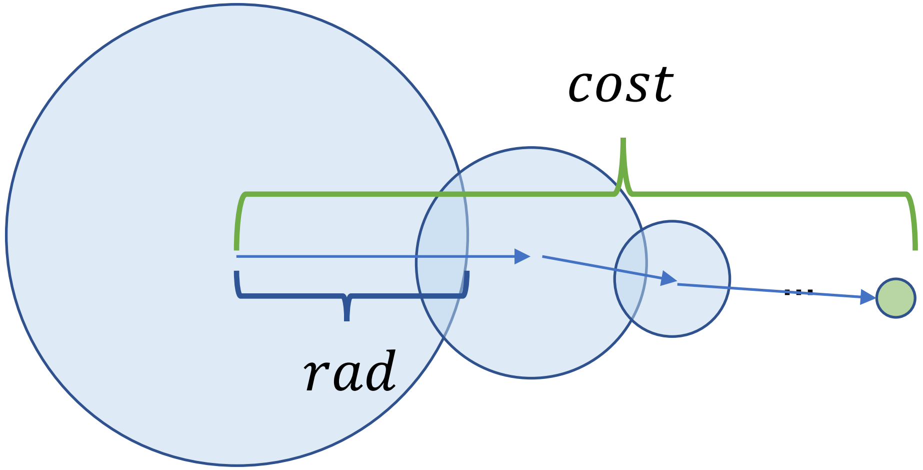


Extra dist. halves at each step

$\Rightarrow \boldsymbol{cost = O(1)\, rad}$

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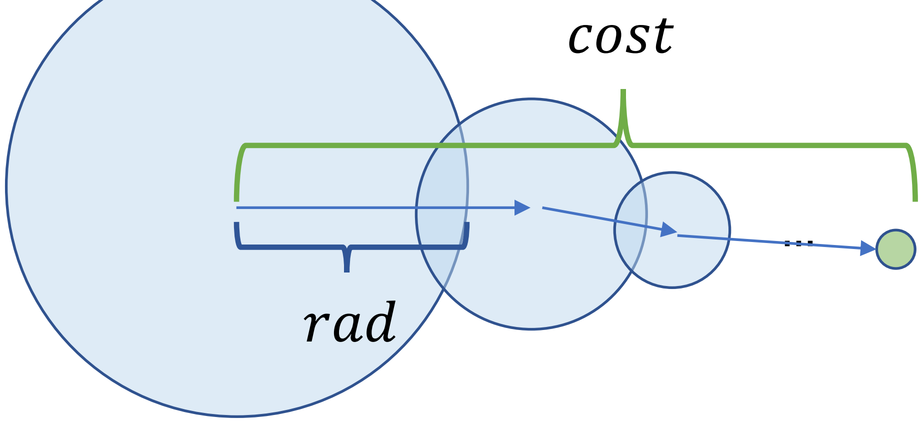


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**Assume:** Client connection cost $= rad$

# Improving KLS: Our Techniques

- **Our Improvement:**
  - Desired Structure = **two sets of** disjoint balls
  - Modify = if **3 balls** intersect ⇒ delete **largest** radius one

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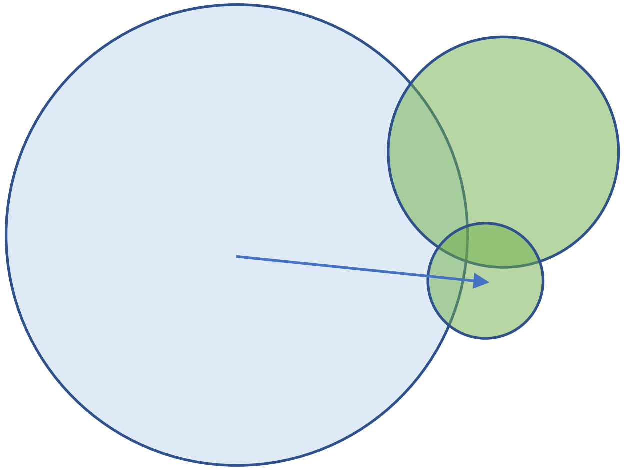

Extra dist. **quarters** at each step

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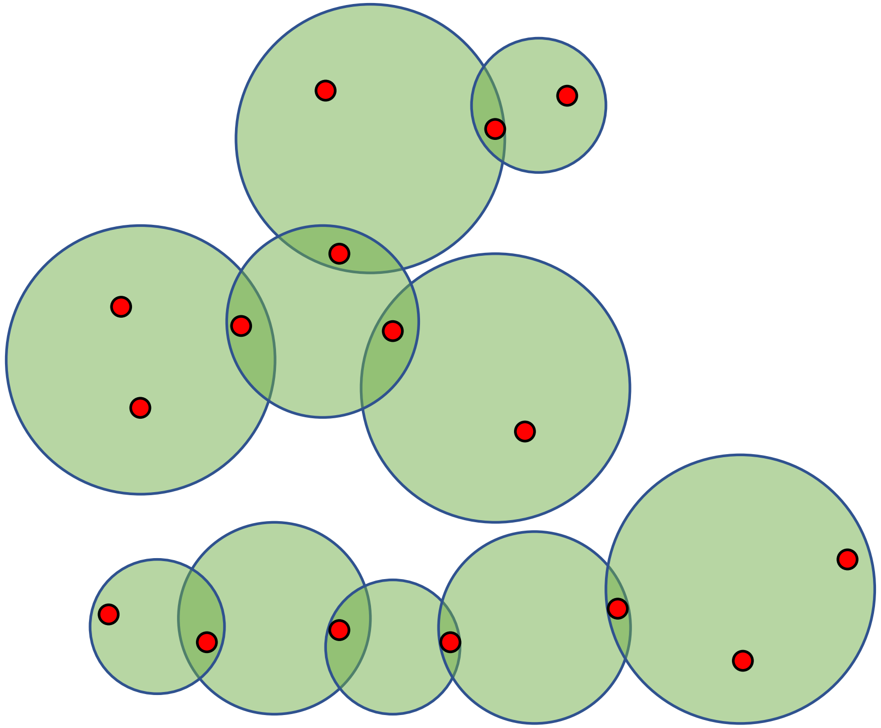

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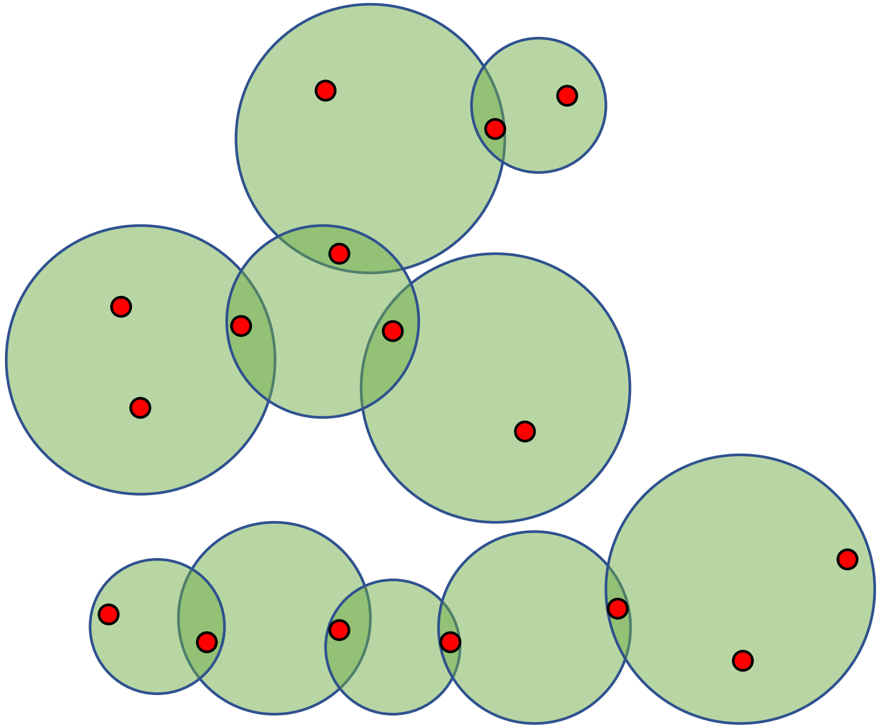
**Remains:** Reduce frac. vars. to $O(1)$



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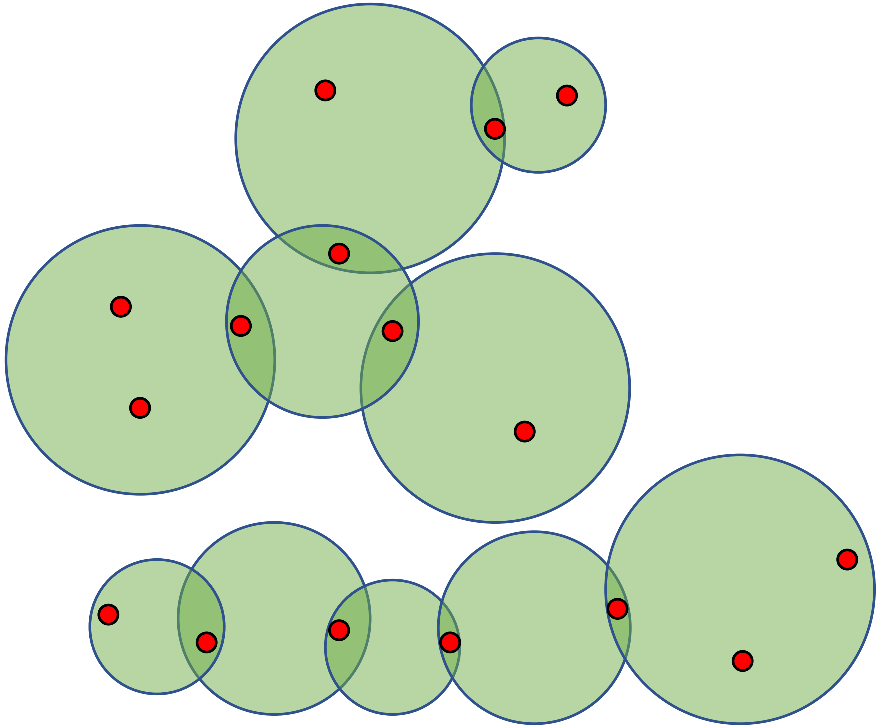
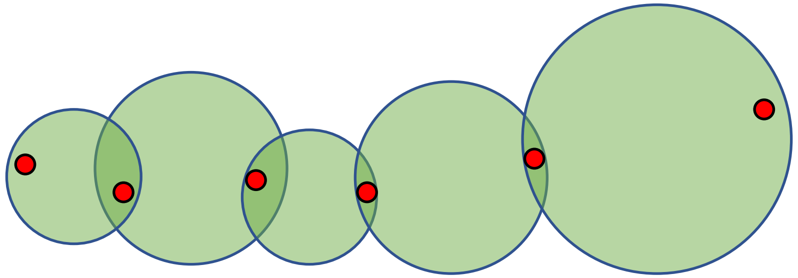


**Remains:** Reduce frac. vars. to $O(1)$

**Main Technique:** Show that extreme points are **highly-structured**; use structure to **further modify** balls

# Structure of Extreme Points

**Theorem (Informal):** The two sets of disjoint balls form **$O(1)$ disjoint 'chains' (chain decomposition)**



$\times O(1)$

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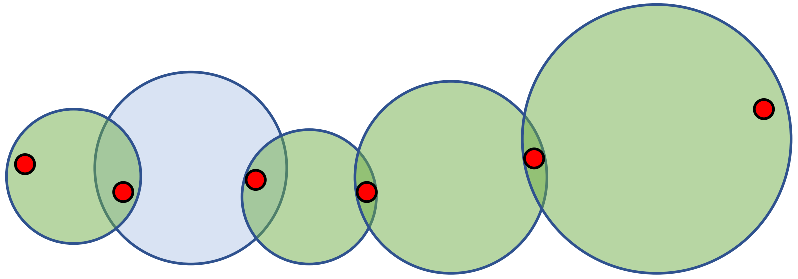

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Use chains to modify balls

# Updated Framework

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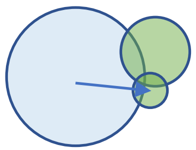

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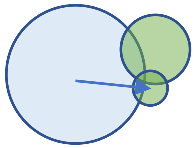

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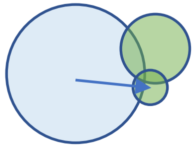


If $O(1)$ frac. vars.

Done

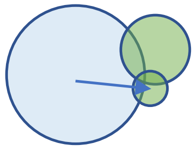
# Updated Framework

Repeat

Else $\Omega(1)$ frac. vars.

**Framework:**
1. Solve LP
2. Modify tight constraints
3. Repeat until tight constraints have desired structure
   $\Rightarrow$ chain decomposition

**To do:** modify tight constraints in chain decomposition

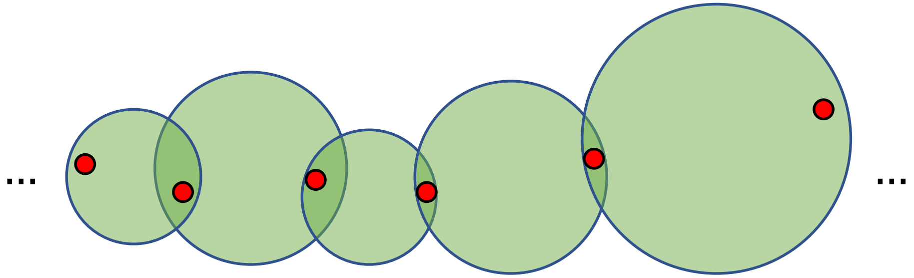


If $O(1)$ frac. vars.

Done

# Using Chain Decompositions

- $\Omega(1)$ frac. vars. $\Rightarrow$ some chain is long



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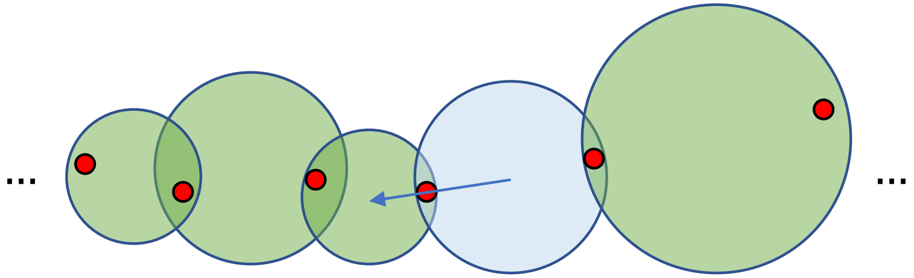

- $\Rightarrow$ delete ball in middle with larger radius

# Updated Framework

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Else $\Omega(1)$ frac. vars.

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   $\Rightarrow$ chain decomposition

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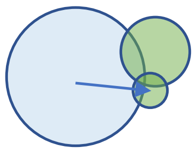

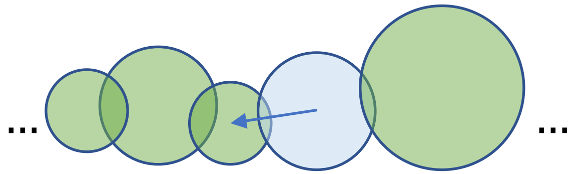

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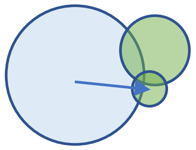
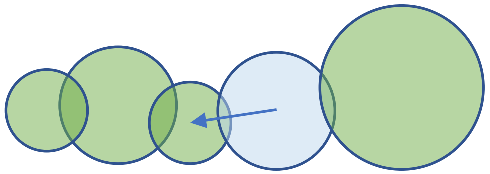
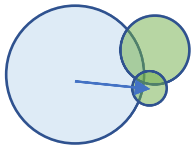
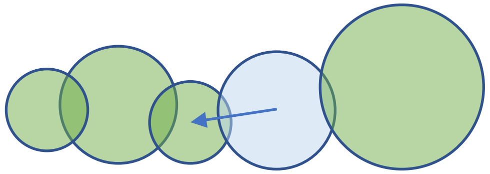
# Final Approx.

- Quarter-steps:  Half-steps: ...  ...

# Final Approx.

**KLS:** only half-steps

- Quarter-steps: 

Half-steps: …  …

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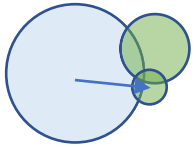
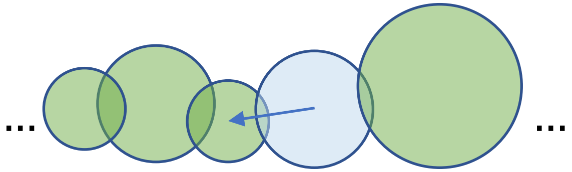
- Quarter-steps: 

Half-steps: 

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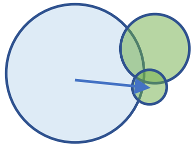
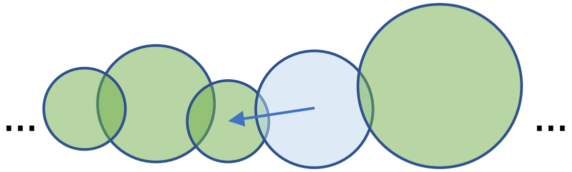
- Quarter-steps: 

Half-steps: 

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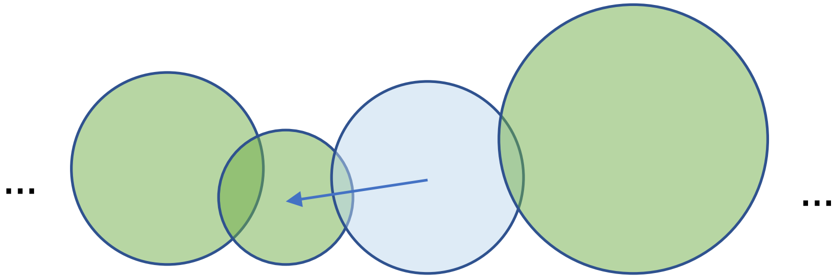

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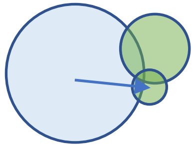
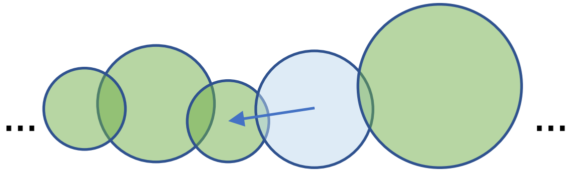
- Quarter-steps: 

Half-steps: 

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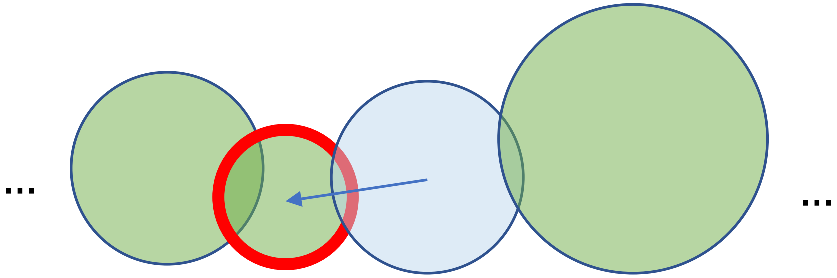

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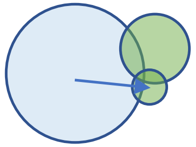
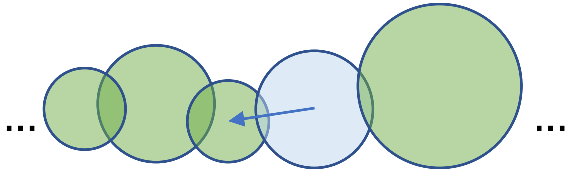
- Quarter-steps: 

Half-steps: 

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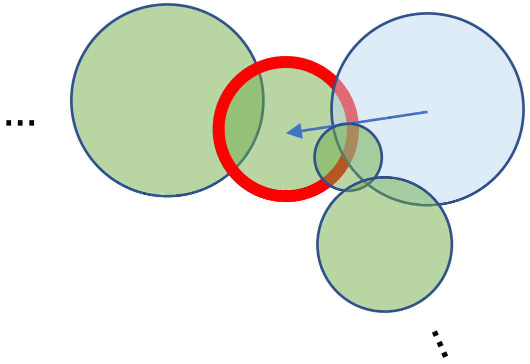

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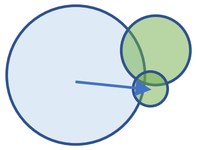
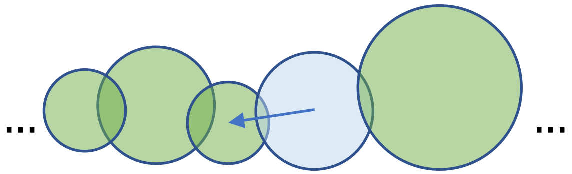
- Quarter-steps: 

Half-steps: 

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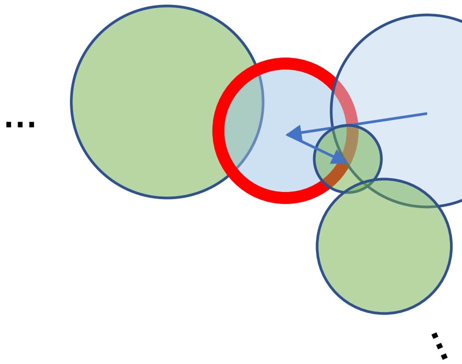

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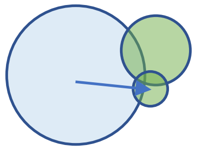
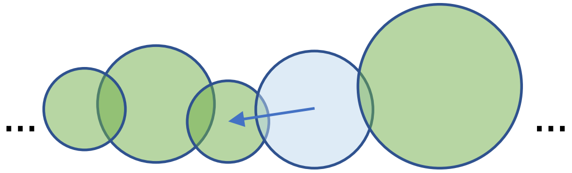
- Quarter-steps: 

Half-steps: 

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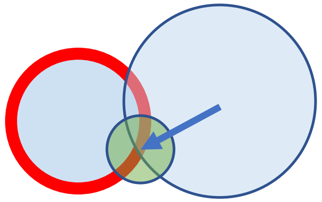

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**Open Questions:**

- Can we allow even richer sets of tight constraints?
- Can we approximate $k$-Median with other side constraints? $O(1)$-many knapsack/coverage constraints?